**Fourier Transforms**

So I want to examine converting expression in position space into (usually simpler) expressions in Fourier space.

**Integral with difference arguments in terms of FT**

Consider the following integral,



We’d like to conjure an equivalent expression in terms of the integrands’ Fourier transforms, defined by (leaving out dot products – x is implicitly a 3D vector, and dx a 3D differential):



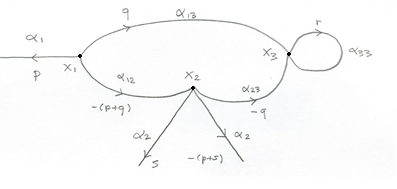
(by the way, the continuum/infinite volume approximation comes from fact that typically wavevectors are given by **q** = 2π(nx/Lx, ny/Ly, nz/Lz), and so each individual wavevector takes up a phase-space volume = (2π(1)/Lx·(2π(1)/Ly·2π(1)/Lz = (2π)3/V. So Σq would translate to ∫dq/[(2π)3/V], which implies (1/V)Σq = ∫d3q/(2π)3) So starting off, we’ll write each term in the integrand in terms of its inverse Fourier transform:



where we use ∫dx ei(q-q´)x = Vδqq´. Call k1 = p, k3 = q, k5 = r, k7 = s. Then we can write:



We can symbolize this with the diagram below:



Examining our diagram leads us to the following conjecture about directly converting our position space expression to Fourier space.

**Rules**

Draw all points in the integral.

For each argument xa – xb, draw a directed line segment from xa to xb.

For each argument xc by itself, draw a directed line segment extending out to empty space. Draw momenta in the direction of each of the arrows, conserving momentum at each vertex Label each of line with the Fourier transform of the function of those two/one arguments.

Multiply them all together and sum, via



over each independent mode.

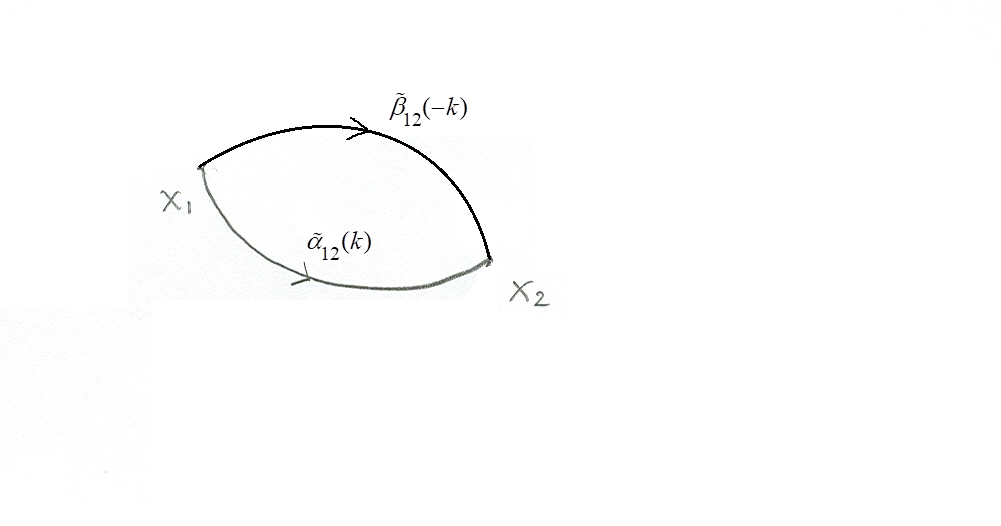
But things aren’t quite that simple. Consider,



Then,



The corresponding diagram would be:



So this follows our expected rules except for the missing factor of 1/V. And that happened because we ended up with the same δ, the ‘momentum conservation law’ on both ends and so I think the lesson here is:

**Rules modification**

If the integration over a point results in the same ‘momentum’ conservation equation as integration over another point, then put a V back in.

**Integral with difference arguments and external points in terms of FT**

We generalize this now to include external points that aren’t integrated over. Consider:



This can be symbolized by the diagram below. Note x4, x5, x6 are all external points, i.e., not integrated over. Well so this is:



where δα means δα0 for short. Continuing,



So we have 7 independent q’s. There are different ways to choose independent variables, and it doesn’t matter how you do it. But for sake of discussion, let:

q01 = k1, q1 = k2, q12 = k3, q2 = k4, q25 = k5, q´2 = k6, q33 = k7

Then the first δ function requires q13 = q01 – q12 – q1 = k1 – k3 – k2.

The second δ function requires q23 = q12 – q­2 - q´2 – q25 = k3 – k4 – k6 – k5.

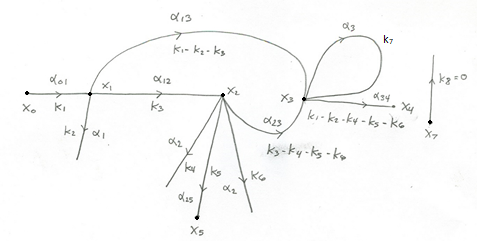
The third delta function requires q34 = q13 + q23 = (k1 – k3 – k2) + (k3 – k4 – k6 – k5) = k1 – k2 – k4 – k5 – k6.

The last delta function requires q7 = 0.

Then we can say,



Can’t say looks better than the position space expression, but usually it will, ‘cause usually we integrate over many more points. Anyway, could symbolize this expression above with following diagram,



We’ll note that we have ‘momentum’ conservation at each integrated-over point. So we might conjecture the following rules:

**Rules**

Draw all points in the integral – including those external, i.e., not integrated over.

For each argument xa - xb, draw a directed line segment from xa to xb.

For each argument xc by itself, draw a directed line segment extending out to empty space. Draw momenta in the direction of each of the arrows, conserving momentum at each vertex *that’s integrated* over.

Label each of lines with the Fourier transform of the function of those two/one arguments.

Each external leg, i.e., a line connecting a point that is integrated over with a point which isn’t, will carry a factor of e±ikx, where x is the external point, k is the momentum along the line, and ± is for momentum heading away from or towards the external point.

Multiply them all together and sum, via



over each independent mode. But put a V back in for every case where a momentum conservation law gets duplicated.

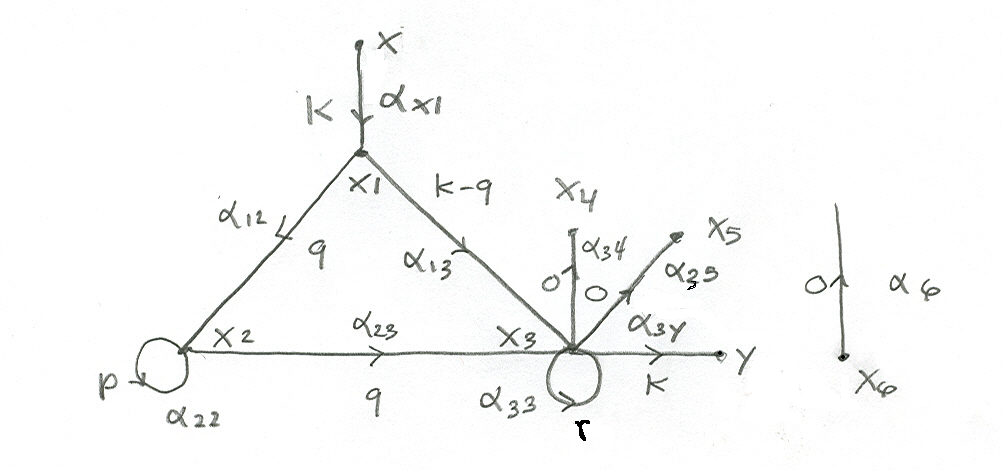
Thus we can rewrite this integral in terms of the Fourier transformed functions as follows.

**FT of integral with difference arguments and just two external points**

Now let’s look at the Fourier transform of a special case of the above, an integral with just two external points. Suppose we have, where x and y are the external points.



Saw from above that we can symbolize the integral itself as, e.g.:



which comes out to:



Note that however we decided to label the momenta, if we have some momentum flowing out from x, then we’d have to have the same amount flowing into y. That’s because integrating over all those internal points fixes momentum conservation, and so whatever flows in must flow out. Now let k´ be the variable w/r to which we take the Fourier transform, so let’s evaluate,



So all we changed was to replace k with k´ and eliminate the external legs factors, and a factor of V. So rules,

**Rules**

Draw all points in the integral – including those external, i.e., not integrated over.

For each argument xa - xb, draw a directed line segment from xa to xb.

For each argument xc by itself, draw a directed line segment extending out to empty space. Draw momenta in the direction of each of the arrows, in particular k´ (or whatever) coming from x and then *necessarily* k´ (or whatever) going into y, conserving momentum at each vertex (because they’re *all* integrated over, except the x and y external points).

Label each of lines with the Fourier transform of the function of those two/one arguments. I

Multiply them all together and sum, via



over each independent mode.

**Integral with external points and sum arguments in terms of FT**

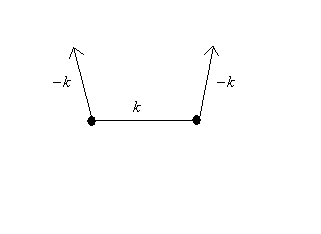
Now suppose that we have something like,



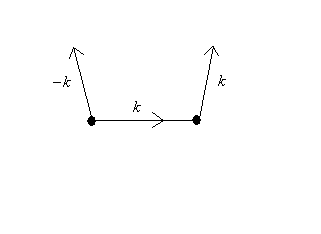
Let’s introduce Fourier transforms for each of the terms and see what we get,



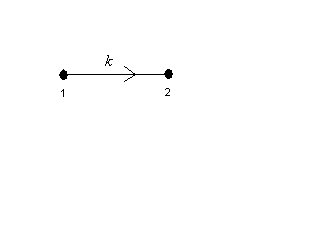
This can be represented as,



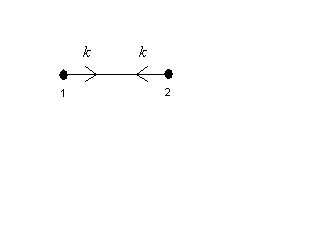
In contrast, if α12 were a difference term, it would’ve come out like,



An obvious characterization is that in the difference argument, momentum is conserved on both ends. But for the sum argument term, momentum is conserved at one end, and anti-conserved at the other, meaning that perhaps a good way to depict the difference term is, as we’ve seen,



And we impose that the momentum coming into each dot is conserved. Perhaps a way to represent the sum term is:



This way works best with external points. Either way, we see that we’re treating the two points equivalently as we ought to. And then we would conserve momentum at each vertex as before. But suppose that we had an external leg. Consider,



Let’s introduce Fourier transforms for each of the terms and see what we get,



So the external leg gets a value of exp(ikx2). In this sense too is the notation is quite informative. For when there is an external leg with momentum pointing away from the point, then the value it is given is: exp(ikx). And when the momentum is proceding towards the external point, then the value of the external leg is exp(-ikx). So we would do as before – label momenta, conserving at each integrated over vertex, label each of the internal and external lines as aforementioned, and sum over all undetermined momenta/V. Let’s consider a more complicated expression:



So again, same thing. But don’t forget that when doing space-time transform, the time arguments are still going to be time differences (if energy is conserved).

**Fourier transforms of simple products**

Consider the product:



It can be represented via their finite Fourier transform as follows.



And the Fourier transform ∫dxe-ikx of the whole is



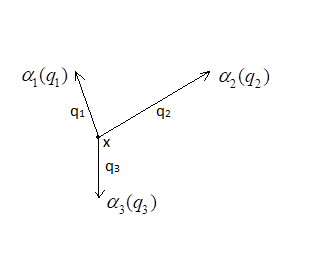
and we could write this as:



so,



which puts it back in the convolution form we often start with…We can represent this in terms of a diagram,



**Rules**

For N order product…

Draw lines extending from x to ∞. Lable the line with momentum qi, and the function’s FT, subject to restriction k (or whatever variable of FT is) = q1 + q2 + … + qN.

Multiply all Fourier transforms together and sum over all (N-1) independent modes via (1/V)Σmode.

**Inverse Fourier transforms of simple products**

Now let’s consider the inverse FT of a product:



which, by subtracting x3 from x2, we can write as:



And so we get the convolution theorem stuff, generalized to multiple functions. It is easier to see this is correct since we already know how to go from RHS to LHS via stuff at top of page.

**Fourier Transform of Special Cases**

Now consider the F.T. of specialized objects



So regardless of the definition of the F.T. function, say fk(x), we may say that,



Now consider,



This holds for the other definition of the transform too. And finally,



and,



So



In a lot of the Condensed Matter stuff, we have a correlation function with technically two independent arguments x, x´, but which we know will ultimately have a simpler functional dependence x - x´.



what is the FT of the thing on the right if it can be expressed as a difference argument? We’d like to know what FTx-x´ is, in terms of FTx{ρ(x)} and FTx´{ρ(x´)}. Well the FT with respect to both arguments is:



so,



switch to difference and COM variables.



and we get,



Now let’s integrate both sides w/r to X.



But there is no X dependence on the LHS and so we just get



So we can say that:



Or something…perhaps we can just do:



**Discrete Fourier Transforms**

Suppose that we have a sum over a translationally invariant lattice. Define:



Then a sum like,



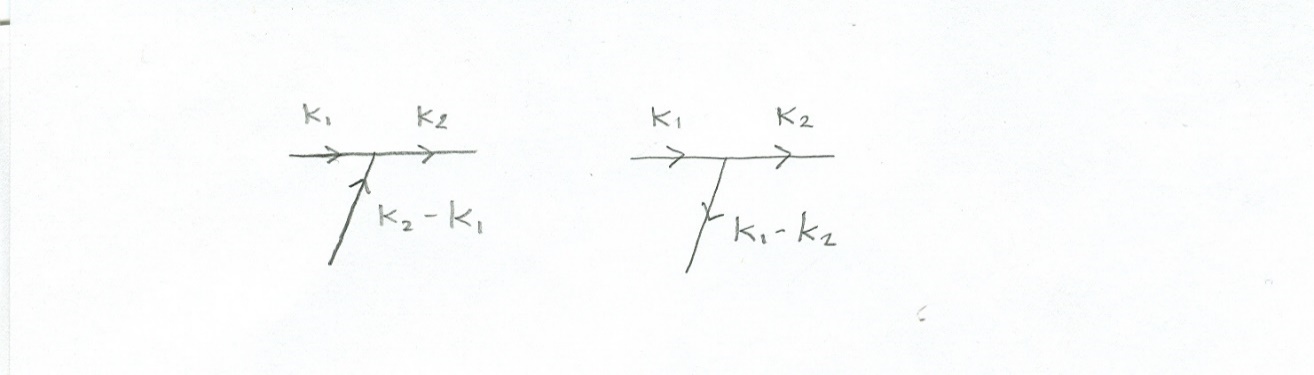
This we would’ve anticipated.

**Rules**

Pretty much the same as continuous transforms, just replacing V with N. External legs would be present as before, with the familiar factor: exp(±ik·Ri) for an external leg that goes into/out of the diagram. We can relate these sums over k to the FT’s via the relation Δk ≈ (2π)3/V.

**A simple momentum conservation reminder**

Keep in mind, when doing diagrams, that:



Think of it as following arrow along the flow line and subtracting the other from that.

**Just checking that we get the usual completeness relation in the discrete case**

So just want to verify, for the discrete case, that:



where the sum over m runs over all N sites of the entire periodic lattice, and **k**-**k**´ is a vector in the Brillouin Zone. Well let’s just make it easier and go to 1D with a lattice of N sites, separated by distance a. Let k-k´ = 2πn/L = 2πn/Na (n≠0), and let Rm = ma. Then,



but if n = 0, then we would’ve gotten,



So it checks out at least in this case.